

UNIFORM EXPONENTIAL GROWTH FOR CAT(0) SQUARE COMPLEXES

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ABSTRACT. In this paper we start the inquiry into proving uniform exponential growth in the context of groups acting on CAT(0) cube complexes. We address free group actions on CAT(0) square complexes and prove the more general statement that if F is a finite collection of hyperbolic automorphisms of a CAT(0) square complex X , then either there exists a pair of elements in $a, b \in F$ and a pair words of length at most 7 in a and b which freely generate a free semigroup, or there exists a flat (of dimension 1 or 2) in X stabilized by all elements of F . As a corollary, we obtain a lower bound for the growth constant, $\sqrt[3]{2}$, which is uniform not just for a given group acting freely on a given CAT(0) cube complex, but for all non virtually abelian groups acting freely on any CAT(0) square complex.

1. INTRODUCTION

Given a group G and a finite generating set S , we let $\mathcal{C}(G, S)$ denote the Cayley graph of G relative to S . The length of an element $g \in G$ with respect to the word metric relative to S is denoted $|g|_S$ and we let $B(S, n)$ denote the ball of radius n in $\mathcal{C}(G, S)$. The *exponential growth rate of G relative to S* is defined to be the following limit (which always exists):

$$\omega(G, S) = \lim_{n \rightarrow \infty} |B(S, n)|^{1/n}.$$

The *exponential growth rate of G* is then given by

$$\omega(G) = \inf\{\omega(G, S) \mid \text{finite generating sets } S\}.$$

The group G is said to have exponential growth if $\omega(G, S) > 1$ for some and therefore for all finite generating sets S . Moreover, G is said to have uniform exponential growth if $\omega(G) > 1$. See de la Harpe [8] for details.

Gromov asked if every group of exponential growth is also of uniform exponential growth. The first example of a group with exponential growth which is not of uniform exponential growth was constructed by Wilson [14]. Wilson's group and future counterexamples were finitely generated. Whether Gromov's question has an affirmative answer for finitely presented groups remains open.

Uniform exponential growth is known to hold for groups with virtually free quotients, hyperbolic groups, soluble groups, linear groups in characteristic zero and groups acting on trees in the sense of Bass Serre theory (see [8] and

references therein). Uniform exponential growth is typically established by constructing free semigroups [1].

Lemma. *Let G be a group. Suppose there exists a constant $C > 0$ such that for any finite generating set S of G , one can find two elements $u, v \in G$ with $\max\{|u|_S, |v|_S\} < C$ and u and v freely generate a free semigroup. Then $\omega(G) \geq \sqrt[3]{2}$*

This method and variations of it often allows one to establish “uniform uniform exponential growth”. Mangahas [11] proved that finitely generated subgroups of the mapping class group $Mod(S)$ of a surface S which are not virtually abelian have uniform exponential growth with minimal growth rate bounded below by a constant depending only, and necessarily, on S . Breuillard [2, Main Theorem] established a different sort of uniformity for linear groups: for every $d \in \mathbb{N}$ there is $N(d) \in \mathbb{N}$ such that if K is any field and F a finite symmetric subset of $GL_d(K)$ containing 1, either $F^{N(d)}$ contains two elements which freely generate a nonabelian free group, or the group generated by F is virtually solvable. We refer the reader to [5] for further examples.

In this paper we start the inquiry into proving uniform exponential growth in the context of groups acting on CAT(0) cube complexes. We address free group actions on CAT(0) square complexes. We do this by proving a more general statement about groups generated by hyperbolic elements.

Theorem 1. *Let F be a finite collection of hyperbolic automorphisms of a CAT(0) square complex. Then either*

- (1) *there exists a pair of elements in $a, b \in F$ and a pair words of length at most 7 in a and b which freely generate a free semigroup, or*
- (2) *there exists a flat (of dimension 1 or 2) in X stabilized by all elements of F .*

As a corollary, we obtain a “uniform uniform” type result, which says that there is a uniform lower bound for growth, not just for a given group, but for all groups acting freely on any CAT(0) square complex.

Corollary 2. *Let G be a finitely generated group acting freely on a CAT(0) square complex. Then either $w(G) \geq \sqrt[3]{2}$ or G is virtually abelian.*

A new class of examples to which the results of this paper apply are the simple groups constructed by Burger and Moses acting on a product of two trees [4]. We expect that a similar result will hold for all dimensions, in that for a finitely generated group G acting freely on a CAT(0) cube complex of dimension n , G will be virtually abelian or $w(G) \geq w_0 > 1$ where, w_0 will depend only on the dimension n , and not on the group or the complex.

2. HYPERPLANES AND GROUP ELEMENTS

We review some relevant basic facts regarding hyperplanes and halfspaces. See, for example, [6] or [12] for more details. We let X be a CAT(0) cube

complex. We use $\mathfrak{h}, \mathfrak{k}$ to denote halfspaces, $\hat{\mathfrak{h}}, \hat{\mathfrak{k}}$ to denote the corresponding hyperplanes and $\mathfrak{h}^*, \mathfrak{k}^*$ to denote the complementary halfspaces.

We let $\text{Aut}(X)$ denote the collection of cubical, inversion-free automorphisms of X . (An inversion is an isometry of X that preserves a hyperplane and inverts the corresponding halfspaces). If G is an action on X which contains inversions, then we may subdivide X so that there are no inversions.

In a CAT(0) cube complex of dimension n , any collection of $n + 1$ hyperplanes contains a disjoint pair. In particular, in the case of a 2-dimensional complex, if $g \in \text{Aut}(X)$ and $\hat{\mathfrak{h}}$ is a hyperplane, then the triple $\{\hat{\mathfrak{h}}, g\hat{\mathfrak{h}}, g^2\hat{\mathfrak{h}}\}$ contains a pair that is either disjoint or equal. Thus, either $g^2\hat{\mathfrak{h}} = \hat{\mathfrak{h}}$, or one of the pair $\{\hat{\mathfrak{h}}, g\hat{\mathfrak{h}}\}, \{\hat{\mathfrak{h}}, g^2\hat{\mathfrak{h}}\}$ is a disjoint pair.

Given a hyperplane $\hat{\mathfrak{h}}$ in X and $g \in \text{Aut}(X)$ a hyperbolic isometry of X , we say that g *skewers* $\hat{\mathfrak{h}}$ if for some choice of halfspace \mathfrak{h} associated to $\hat{\mathfrak{h}}$, we have $g^2\mathfrak{h} \subset \mathfrak{h}$. This property is equivalent to saying that any axis for g intersects $\hat{\mathfrak{h}}$ in a single point.

We say that a hyperbolic isometry $g \in \text{Aut}(X)$ is *parallel* to $\hat{\mathfrak{h}}$ if any axis for g is a bounded distance from $\hat{\mathfrak{h}}$.

We say that a hyperbolic isometry is *peripheral* to $\hat{\mathfrak{h}}$ if it neither skewers $\hat{\mathfrak{h}}$ nor is parallel to $\hat{\mathfrak{h}}$. In this case, we have that any axis lies in a halfspace \mathfrak{h} bounded by the hyperplane $\hat{\mathfrak{h}}$ and is not contained in any neighborhood of $\hat{\mathfrak{h}}$. It follows that either $g\mathfrak{h}^* \subset \mathfrak{h}$ or $g^2\mathfrak{h}^* \subset \mathfrak{h}$.

Definition 3. Given a hyperbolic isometry $g \in \text{Aut}(X)$, we define the skewer set of g , denoted $sk(g)$, as the collection of all hyperplanes skewered by g . We define a disjoint skewer set for g as a collection of disjoint hyperplanes in $sk(g)$ which is invariant under g^2 .

Note that if g is parallel to a hyperplane $\hat{\mathfrak{h}}$, then any hyperplane in $sk(g)$ intersects $\hat{\mathfrak{h}}$. Since there are no intersecting triples of hyperplanes in X , this means that no two hyperplanes in $sk(g)$ intersect. Furthermore, any two translates of $\hat{\mathfrak{h}}$ under $\langle g \rangle$ are parallel to g and hence cross every hyperplane in $sk(g)$. Again, by the two dimensionality of X , this means that the two translates of $\hat{\mathfrak{h}}$ under $\langle g \rangle$ are disjoint. We record this observation, since we will make use of it.

Observation 4. If g is parallel to $\hat{\mathfrak{h}}$, then all the hyperplanes in $sk(g)$ are disjoint and two distinct hyperplanes in the orbit of $\hat{\mathfrak{h}}$ under $\langle g \rangle$ are disjoint.

Lemma 5. Let g be a hyperbolic automorphism of X , then $sk(g)$ is a union of finitely many disjoint skewer sets.

Proof. Consider $\hat{\mathfrak{h}} \in sk(g)$. If $g\hat{\mathfrak{h}} \cap \hat{\mathfrak{h}} = \emptyset$, we let $P_1 = \{g^n(\hat{\mathfrak{h}}) | n \in \mathbf{Z}\}$. Otherwise, since X is 2-dimensional, we have $g^2\hat{\mathfrak{h}} \cap \hat{\mathfrak{h}} = \emptyset$. We then set $P_1 = \{g^{2n}(\hat{\mathfrak{h}}) | n \in \mathbf{Z}\}$ and $P_2 = \{g^{2n+1}\hat{\mathfrak{h}} | n \in \mathbf{Z}\}$. Thus P_1 and P_2 break up the orbit of $\hat{\mathfrak{h}}$ under $\langle g \rangle$ into two disjoint skewer sets. Since there are

finitely many orbits of hyperplanes in $sk(g)$ under the action of $\langle g \rangle$, this breaks up $sk(g)$ into finitely many disjoint skewer sets. \square

Example 6. Let X denote the Euclidean plane, squared in the usual way by unit squares. Let g be an integer translation in the vertical direction. Then the skewer set of g is the collection of horizontal hyperplanes and the number of disjoint skewer sets depends on the translation length of g .

Example 7. Again, let X denote the Euclidean plane. Let g be a glide reflection along the diagonal axis: $g(x, y) = (y + 1, x + 1)$. Then the skewer set of g is a union of four disjoint skewer sets, each invariant under g^2 .

3. THE PARALLEL SUBSET OF AN ELEMENT

Given a hyperbolic $g \in \text{Aut}(X)$, we describe combinatorially a certain invariant subcomplex associated to g which consists of all the lines parallel to axes in G . (This subcomplex is discussed as well in [10] and is slightly different than the minimal set of G , as described in [3] or [9].)

We consider the following partition of hyperplanes $\hat{\mathcal{H}}$ of X . Let

$$\hat{\mathcal{H}}_{\parallel}(g) = \{\hat{h} | \hat{h} \text{ intersects every hyperplane in } sk(g)\}$$

$$\hat{\mathcal{H}}_P(g) = \hat{\mathcal{H}} - (sk(g) \cup \hat{\mathcal{H}}_{\parallel}(g))$$

Since the elements of $\hat{\mathcal{H}}_P(g)$ are peripheral to g , it follows that for each hyperplane $\hat{h} \in \hat{\mathcal{H}}_P(g)$, there exists a well-defined halfspace \hat{h} containing all the axes of g . Recall that the collection of cubes intersecting a hyperplane \hat{h} has a product structure $\hat{h} \times [0, 1]$. We let $N(\hat{h}) = \hat{h} \times (0, 1)$. For a halfspace \hat{h} we let $R(\hat{h}) = \hat{h} - N(\hat{h})$.

We define

$$Y_g = \bigcap_{\ell_g \in \hat{h} \text{ and } \hat{h} \in \hat{\mathcal{H}}_P(g)} R(\hat{h})$$

The subspace Y_g is a $\langle g \rangle$ -invariant convex subcomplex of X , and as Y_g contains the axes of g , it is non-empty.

The hyperplanes intersecting Y_g are the hyperplanes of $sk(g)$ and $\hat{\mathcal{H}}_{\parallel}(g)$. Since $sk(g)$ and $\hat{\mathcal{H}}_{\parallel}(g)$ are transverse collections of hyperplanes, we obtain (by [6]) that Y_g admits a product structure $Y_g \cong E_g \times T_g$, where E_g is defined by the hyperplanes $sk(g)$ and T_g is defined by the hyperplanes in $\hat{\mathcal{H}}_{\parallel}(g)$. Note that $sk(g)$ does not contain any disjoint facing triples of hyperplanes. As g does not skewer any hyperplane in $\hat{\mathcal{H}}_{\parallel}(g)$, g fixes a vertex in T_g . Since Y_g is 2-dimensional, there are two possibilities:

- (1) $E_g = \mathbf{R}$ and T_g is isomorphic to a tree.
- (2) E_g is 2-dimensional and T_g is a point.

We call Y_g the *parallel set* of g and E_g its *Euclidean factor*.

We need a further understanding of E_g in order to conclude that groups that stabilize it have nice properties.

Lemma 8. *Let E_g be the Euclidean factor of Y_g . Then either E_g is a Euclidean plane or E_g contains an $\text{Aut}(E_g)$ -invariant line.*

Proof. Note that $\langle g \rangle$ acts on E_g with finitely many orbits of hyperplanes. Consider one such hyperplane \hat{h} . Note that by the fact that E_g does not contain facing triples of hyperplanes, it follows that \hat{h} is a tree with vertices of valence 1 or 2. From this it follows that the hyperplanes in E_g are either lines, rays or closed intervals.

Suppose some hyperplane $\hat{h} \in E_g$ is a line. Then we have a disjoint skewer set $\hat{\mathcal{K}} = \{g^{2n}\hat{h} | n \in \mathbf{Z}\}$ for g . Since there are no facing triples, we have that all hyperplanes between hyperplanes in $\hat{\mathcal{K}}$ are also lines. (See Figure 1.)

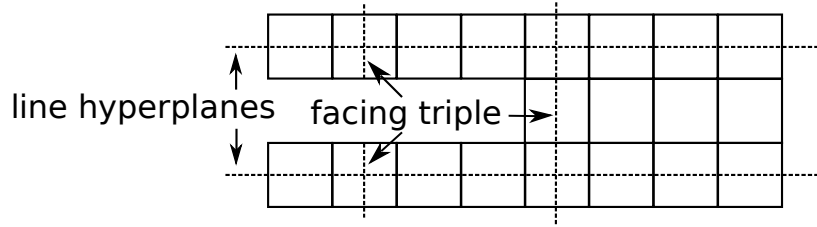


FIGURE 1. A facing triple of hyperplanes in E_g if there are some line hyperplanes, but not all are line hyperplanes.

Thus, we may assume none of the hyperplanes are lines. It follows that all hyperplanes are either rays or closed intervals. If there exists a hyperplane in E_g which is a closed interval, then by similar considerations as above, we may conclude that all hyperplanes are closed intervals. Since $\langle g \rangle$ acts cocompactly on E_g , it follows all lines in E_g are parallel and the space of such lines is a compact interval I . Since the action of $\text{Aut}(E_g)$ on I has a fixed point, it then follows that there is an $\text{Aut}(E_g)$ -invariant line.

Finally, we consider the case in which all hyperplanes are rays. In this case, E_g is a "staircase", as in Figure 2. The space of lines which coarsely contains the endpoints of the hyperplanes is itself a ray R which is $\text{Aut}(E_g)$ -invariant, hence there is an $\text{Aut}(E_g)$ fixed point in R and hence an $\text{Aut}(E_g)$ -invariant line in E_g .

□

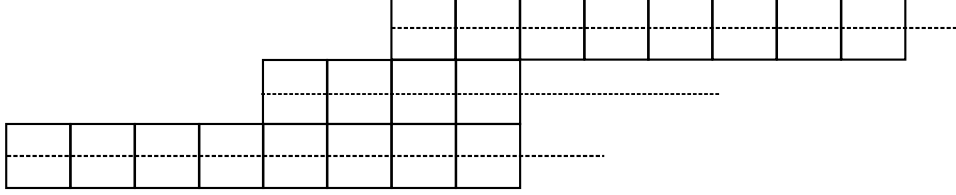


FIGURE 2. The case in which all hyperplanes in E_g are rays. The endpoints of the rays are invariant, and hence any line in E_g a bounded distance from all endpoints is $\text{Aut}(E_g)$ -invariant

4. THE PING-PONG LEMMA AND HYPERPLANE PATTERNS THAT YIELD FREE SEMIGROUPS

We will use the following version of the Ping Pong Lemma (see, for example, [7])

Lemma 9 (Semigroup Ping Pong). *Suppose that a group G is acting on a set X and U, V are disjoint subsets of X . The elements $a, b \in G \setminus \{1\}$ satisfy*

- $a(U \cup V) \subset U$
- $b(U \cup V) \subset V$

Then a and b freely generate a free subsemigroup in G .

Proof. Let Σ be the semigroup generated by a and b in G . Observe that for any $g, h \in \Sigma \subset G$, $ag = ah$ or $bg = bh$ in Σ if and only if $g = h$ in Σ . Therefore, it is enough to check that two words of the form ag and bh cannot be equal in Σ . But, $ag(U \cup V) \subset U$ and $bh(U \cup V) \subset V$. Since, $U \cap V = \emptyset$, $ag \neq bh$. \square

4.1. On groups acting on trees. To warm up, and to record a few observations we use later on, we first explore what happens for a pair of hyperbolic isometries acting on a tree. We include the proofs here because we will need these type of arguments. However, this is not new. See, for example, [1]. Let T be a simplicial tree. Recall if an element g of $\text{Aut}(T)$ is hyperbolic then there is a unique geodesic ℓ_g (called the axis of g) which is invariant under g on which g induces a translation.

Proposition 10. *If a and b are two hyperbolic automorphisms of a tree T , then one of the following occurs:*

- a, b share the same axis,
- $a^{\pm 1}$ and $b^{\pm 1}$ freely generate a free semigroup.

Proof. Suppose that $\ell_a \neq \ell_b$. First assume that $\ell_a \cap \ell_b$ contains an edge $e = [p, q]$. (See Figure 3.) Choose e so that q is a point of bifurcation of ℓ_a and ℓ_b . Let T_q be the component of $T - \text{interior}(e)$ containing q . After possibly replacing a by a^{-1} and/or b by b^{-1} , we see that $ae \subset T_q$ and $be \subset T_q$. Set $U = aT_q$ and $V = bT_q$. Then U, V satisfy the hypothesis of Lemma 9. We will generalize this argument in our context,

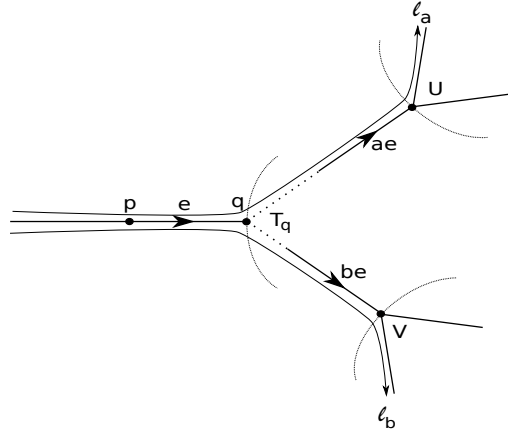


FIGURE 3. The hyperbolic isometries a and b have non-equal, but overlapping axes.

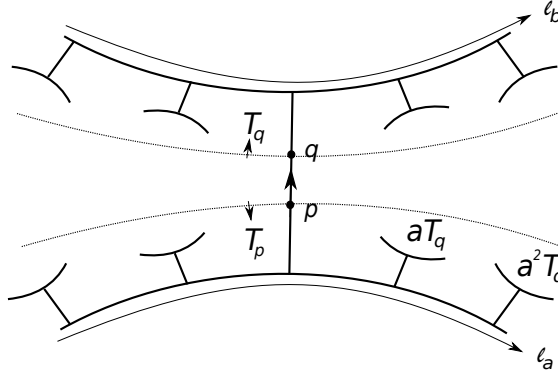


FIGURE 4. The hyperbolic isometries a and b have disjoint axes.

In the case that $\ell_a \cap \ell_b = \emptyset$, we make a different argument, which we will also generalize. (See Figure 4.) Consider an edge $e = [p, q]$ situated along the

geodesic arc joining ℓ_a and ℓ_b . Let T_p be the component of $T - \text{interior}(e)$ containing p and T_q be the component of $T - \text{interior}(e)$ containing q . Suppose (without loss of generality) that $\ell_a \subset T_p$ and $\ell_b \subset T_q$. Then letting $U = \bigcup_{n>0} a^n T_q$ and $V = \bigcup_{n>0} b^n T_p$, we see that $a(U \cup V) \subset U$ and $b(U \cup V) \subset V$, as required. In fact, in this case, we can argue that a and b generate a free group by adjusting U and V to include all non-zero powers of a and b , but we will not need this fact. Note that there is a singular case in which ℓ_a and ℓ_b intersect in a single point. In this case, we simply use the intersecting vertex to separate T into two subtrees, each containing a different axis, and proceed in the same manner. \square

4.2. Back to CAT(0) cube complexes. The following two Lemmas work in any dimension and so in this section, we let X be an n -dimensional CAT(0) cube complex.

Lemma 11. *Let $g_1, g_2 \in \text{Aut}(X)$ and suppose that there exists a halfspace \mathfrak{h} of X such that $g_1 \mathfrak{h} \subset \mathfrak{h}$ and $g_2 \mathfrak{h} \subset \mathfrak{h}^*$. Then g_1, g_2 generate a free semigroup.*

Proof. This argument resembles the first case in the proof of Proposition 10. Set $U = g_1 \mathfrak{h}$ and set $V = g_2 \mathfrak{h}$ and apply Lemma 9 \square

We call the triple $\{\mathfrak{h}, g_1 \mathfrak{h}, g_2 \mathfrak{h}\}$ a *ping pong triple* for g_1 and g_2 .

Lemma 12. *Let $\hat{\mathfrak{h}}$ be a hyperplane of X . Suppose that there exist two hyperbolic elements $g_1, g_2 \in \text{Aut}(X)$ such that*

- *there exists an axis l_1 for g_1 such that $l_1 \subset \hat{\mathfrak{h}}$*
- *there exists an axis l_2 for g_2 such that $l_2 \subset \hat{\mathfrak{h}}^*$*
- *both g_1 and g_2 are peripheral with respect to $\hat{\mathfrak{h}}$.*

Then there exist $\epsilon_1, \epsilon_2 \in \{1, \dots, n\}$ such that $g_1^{\epsilon_1}$ and $g_2^{\epsilon_2}$ freely generate a free semigroup.

Proof. This argument resembles the second case of the proof of Proposition 10. As both g_1 and g_2 are peripheral with respect to $\hat{\mathfrak{h}}$, there exist $\epsilon_1, \epsilon_2 \in \{1, \dots, n\}$ such that $g_1^{\epsilon_1} \hat{\mathfrak{h}} \cap \hat{\mathfrak{h}} = \emptyset$ and $g_2^{\epsilon_2} \hat{\mathfrak{h}} \cap \hat{\mathfrak{h}} = \emptyset$. Moreover, $g_1^{\epsilon_1} \mathfrak{h}^* \subset \mathfrak{h}$ and $g_2^{\epsilon_2} \mathfrak{h} \subset \mathfrak{h}^*$. Set

$$S_1 = \bigcup_{n>0} g_1^{n\epsilon_1} \mathfrak{h}^*, \quad S_2 = \bigcup_{n>0} g_2^{n\epsilon_2} \mathfrak{h}$$

The sets S_1, S_2 do not intersect one another. As $S_1 \subset \mathfrak{h}$, we see that $g_2^{n\epsilon_2} S_1 \subset S_2$. Similarly, we have $g_1^{n\epsilon_1} S_2 \subset S_1$. We now apply the Ping Pong Lemma stated earlier to deduce that $g_1^{\epsilon_1}$ and $g_2^{\epsilon_2}$ freely generate a free subsemigroup in G . \square

We remark here that using a well-known group theoretic version of the Ping Pong Lemma, one can in fact show that $g_1^{\epsilon_1}$ and $g_2^{\epsilon_2}$ freely generate a free subgroup in G .

5. MAIN ARGUMENT

Now, let X be a CAT(0) square complex.

Lemma 13 (All or nothing). *Let a and b be hyperbolic isometries of X and let P be a disjoint skewer set for a . Suppose that no pair of words of length at most 6 in a and b generate a free semigroup, then either b skewers every hyperplane in P or b does not skewer any hyperplane in P .*

Proof. Recall that for any \hat{h} in $sk(a)$, there exists an associated halfspace \hat{h} such that $a^2\hat{h} \subset \hat{h}$. If b skewers some element in P , but not all, we may also choose \hat{h} such that \hat{h} is skewered by b but $a^2\hat{h}$ is not skewered by b . After replacing b possibly by b^{-1} , we may assume that $b^2\hat{h} \subset \hat{h}$. Note that b and hence b^2 is peripheral to $a^2\hat{h}$.

Now by the 2-dimensionality of X , either $b^2a^2\hat{h} \cap a^2\hat{h} = \emptyset$ or $b^4a^2\hat{h} \cap a^2\hat{h} = \emptyset$. We further have that $b^2a^2\hat{h} \subset b^2\hat{h} \subset \hat{h}$ and $b^4a^2\hat{h} \subset b^4\hat{h} \subset \hat{h}$.

We thus have that either $\{\hat{h}, a^2\hat{h}, b^2a^2\hat{h}\}$ or $\{\hat{h}, a^2\hat{h}, b^4a^2\hat{h}\}$ is a ping pong triple of halfspaces for the pairs a^2, b^2a^2 or a^2, b^4a^2 . In either case, we obtain words of length at most 6 freely generating a free semigroup, a contradiction.

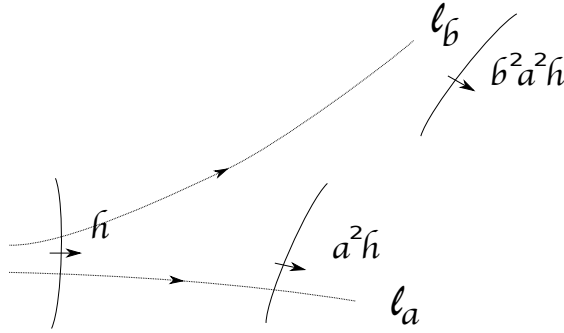


FIGURE 5. The element b skewering \hat{h} but not $a\hat{h}$.

□

Proposition 14 (Not skewering means parallel). *Let a and b be hyperbolic isometries of X and let P be a disjoint skewer set for a . Let ℓ_b be an axis for b . Suppose that b does not skewer any element of P and that no pair of words of length no more than 7 freely generate a free semigroup. Then*

- (1) *the axis ℓ_b is parallel to every hyperplane $\hat{h} \in P$.*
- (2) *$bP \in sk(a)$*
- (3) *b^2 stabilizes every hyperplane in P .*

Proof. By assumption, there exists $h \in P$ such that $a^2 h \subset h$ and $P = \{a^{2k} \hat{h} \mid k \in \mathbf{Z}\}$. We set $c = a^2$. Since b does not skewer any hyperplane in P , we may assume that $\ell_b \subset h \cap c h^*$. (We are using here that the action is without inversions, so that if $\ell_b \subset \hat{h}$ for some hyperplane, there is a parallel axis for b on either side of \hat{h} .)

We will now use our assumptions to remove the possibility that b is peripheral to \hat{h} or $c \hat{h}$.

First, suppose b is peripheral to both \hat{h} and $c \hat{h}$. In this case, we note that $b^c = c^{-1} b c$ is an element whose axis lies in h^* and is peripheral to \hat{h} . We can then apply Lemma 12 to conclude that for some $\epsilon_1, \epsilon_2 \in \{1, 2\}$, the elements b^{ϵ_1} and $(c^{-1} b c)^{\epsilon_2}$ generate a free semigroup. These are words of length at most 7 in a and b .

Let us now assume that b is parallel to \hat{h} but peripheral to $c \hat{h}$.

It follows from Observation 4 that for any $i \in \mathbf{Z}$, $b^i \hat{h} = \hat{h}$ or $b^i \hat{h} \cap \hat{h} = \emptyset$.

First let us consider the case that $b^2 \hat{h} = \hat{h}$. Note that since we are assuming that $\text{Aut}(X)$ acts with no inversions, we have that $b^2 \hat{h} = \hat{h}$. Now since b is peripheral to $c \hat{h}$, for $k = 1$ or 2 , we have that $b^{2k} c \hat{h} \cap c \hat{h} = \emptyset$. We thus obtain a ping pong triple of halfspaces $\{\hat{h}, c \hat{h}, b^{2k} c \hat{h}\}$ for the elements c and $b^{2k} c$. From Lemma 11 we see that c and $b^{2k} c$ freely generate a free semigroup, a contradiction since these are words of length at most 6 in a and b . (See Figure 6.)

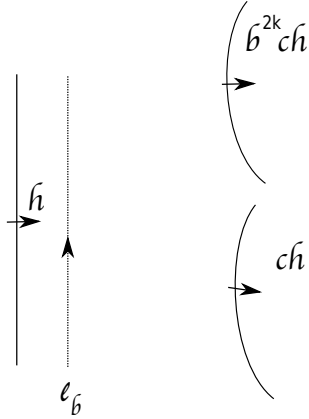


FIGURE 6. If b stabilizes h , we obtain a ping-pong triple of hyperplanes.

We may thus assume that $b \hat{h} \cap \hat{h} = \emptyset$ and $b^2 \hat{h} \cap \hat{h} = \emptyset$. Only one of $b \hat{h}, b^2 \hat{h}$ can separate \hat{h} and $c \hat{h}$, for otherwise we would have $b \hat{h} \subset b^2 \hat{h}$ or $b^2 \hat{h} \subset b \hat{h}$. So for some $\epsilon = 1$ or 2 , we can assume that $b^\epsilon \hat{h}$ does not separate \hat{h} and $c \hat{h}$. Note also that since $c \hat{h}$ is peripheral to b , one cannot have $b \hat{h} \subset c \hat{h}$.

If $c\hat{h} \cap b^\epsilon \hat{h} = \emptyset$, then we obtain a ping-pong triple of halfspace $\{c\hat{h}^*, \hat{h}^*, b^\epsilon \hat{h}^*\}$ for the words c^{-1} and $b^\epsilon c^{-1}$. Since these are words of length at most 4 in a and b , we have a contradiction. (See Figure 7.)

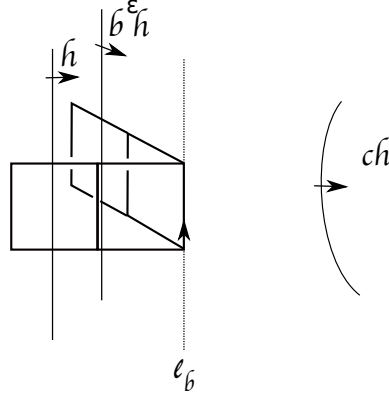


FIGURE 7. If $c\hat{h} \cap b^\epsilon \hat{h} = \emptyset$ and $b^\epsilon \hat{h}$ does not separate \hat{h} and $c\hat{h}$, we obtain a ping pong triple

Thus we assume that $b^\epsilon \hat{h} \cap c\hat{h} \neq \emptyset$ and refer to Figure 8.

Since, by Observation 4, any hyperplane in $sk(b)$ intersects $b\hat{h}$, and we are assuming that $b\hat{h} \cap c\hat{h} \neq \emptyset$, the 2-dimensionality of X implies that any hyperplane in $sk(b)$ is disjoint from $c\hat{h}$. Moreover, by Observation 4, we have that for any hyperplane \hat{k} in $sk(b)$, $b\hat{k} \subset k$ for some choice of halfspace κ associated to \hat{k} . We may further choose κ such that $c\hat{h} \subset \kappa \cap b\hat{k}^*$.

Applying c^{-1} , we see that $\hat{h} \subset c^{-1}\kappa \cap c^{-1}b\hat{k}^*$. Applying b^ϵ , we now see that $b^\epsilon c^{-1}\hat{k} \subset b^\epsilon \hat{h}^* \subset \hat{h}$. Thus we have a ping pong triple of half spaces $\{c^{-1}b\hat{k}^*, c^{-1}\hat{k}^*, b^\epsilon c^{-1}\hat{k}^*\}$ for the elements $c^{-1}b^{-1}c$ and $b^\epsilon c^{-1}b^{-1}c$. So by Lemma 9 we have that $c^{-1}b^{-1}c$ and $b^\epsilon c^{-1}b^{-1}c$ generate a free semigroup and these are words of length at most 7.

We may thus assume that b is parallel to both \hat{h} and $c\hat{h}$. Assume, that $d(\ell_b, \hat{h}) \leq d(\ell_b, c\hat{h})$. (There is no loss of generality here, for if $d(\ell_b, c\hat{h}) \leq d(\ell_b, \hat{h})$, we will reverse the roles of \hat{h} and $c\hat{h}$ in the following argument.)

As before, we first consider what happens if \hat{h} is not stabilized by b^2 . Here we obtain $\hat{h}, b\hat{h}$ and $b^2\hat{h}$ are disjoint. We cannot have that $b\hat{h} = c\hat{h}$ or $b^2\hat{h} = c\hat{h}$, for then we would obtain $c^{-1}b\hat{h}$ or $c^{-1}b^2\hat{h}$ is an inversion of \hat{h} . Thus, we have that $b\hat{h} \subset c\hat{h}^*$ and $b^2\hat{h} \subset c\hat{h}^*$. We now proceed as in the case in which $c\hat{h}$ is peripheral to b to produce a ping pong triple of halfspaces $\{c\hat{h}^*, \hat{h}^*, b^\epsilon \hat{h}^*\}$ for the words c^{-1} and $b^\epsilon c^{-1}$. (The configuration is the same as in Figure 7 except that here $c\hat{h}$ is parallel to ℓ_b .)

So assume $b^2\hat{h} = \hat{h}$. Again, as above, if b^2 did not also stabilize $c\hat{h}$, we would obtain a small ping pong triple. Thus b^2 stabilized $c\hat{h}$ as well. Since b^2 stabilizes $c\hat{h}$ (and the action is inversion-free), we have an axis for b^2 in

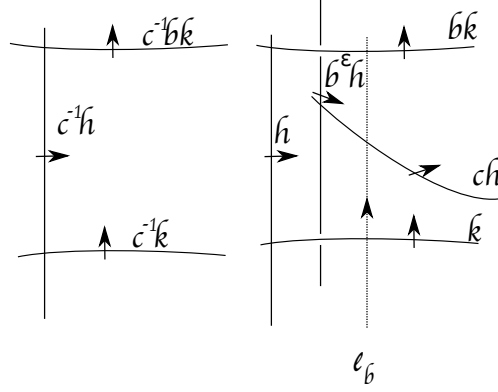


FIGURE 8. If $c\hat{h} \cap b^\epsilon \hat{h} \neq \emptyset$, we obtain a ping pong triple.

$c\hat{h} \cap c^2\hat{h}^*$. We can now carry out all the above arguments replacing \hat{h} and $c\hat{h}$ with $c\hat{h}$ and $c^2\hat{h}$ to conclude that b^2 stabilizes $c^2\hat{h}$. Proceeding in this way we see that b is parallel to every hyperplane of P and that $b^2P = P$.

We are left to show that $bP \subset sk(a)$. By Lemma 13, we have that a skewers every hyperplane in bP or none of the hyperplanes in bP . In the former case, we get $bP \subset sk(a)$ as required. So suppose that a does not skewer any hyperplane in bP . Since $\ell_b \subset \hat{h} \cap c\hat{h}^*$, we have either $b\hat{h}^* \subset \hat{h} \cap c\hat{h}^*$ or $b\hat{h} \subset \hat{h} \cap c\hat{h}^*$, depending on which of \hat{h} or $c\hat{h}$ is closer to ℓ_b . In either case, we then get a small ping pong triple, a contradiction. \square

If a and b are elements such that there exists a disjoint skewer set P for a as in Proposition 14, then we say that b is *subparallel* to a .

Corollary 15. *Given hyperbolic isometries a and b that such that no words of length at most 7 generate a free semigroup, b is subparallel to a if and only if $sk(a) - sk(b) \neq \emptyset$.*

Proof. If b is subparallel to a , then by definition, there exists a disjoint skewer set for a such that b is parallel to all the hyperlanes in P . Thus $P \subset sk(a) - sk(b)$. Conversely, if there exists $\hat{h} \in sk(a) - sk(b)$, then by Lemma 13, the entire disjoint parallel set P for a containing \hat{h} is not skewered by b . Then by Proposition 14, b is subparallel to a . \square

From this corollary, we see that there are three possibilities for two hyperbolic elements a and b so that words of length at most 7 do not freely generate a free semigroup.

- I $sk(a) = sk(b)$
- II b is subparallel to a and a is subparallel to b
- III b is subparallel to a and a is not subparallel to b (or the same with the roles of a and b reversed)

We claim that in each of these cases, we can find an invariant line or flat for $\langle a, b \rangle$.

Proposition 16. *Let a and b be hyperbolic isometries such that no words in a and b of length at most 7 freely generate a free semigroup, then there exists a Euclidean subcomplex of X invariant under $\langle a, b \rangle$.*

Proof. We analyze the three cases above. Suppose we are in Case I, so that $sk(a) = sk(b)$. Then we consider $Y = Y_a = Y_b = E \times T$. If T is trivial (i.e a single point), then we have that both a and b leave E invariant, as required. Otherwise we have that $Y = \mathbf{R} \times T$, where a and b both act by vertical translation. We consider the action of a and b on T . Both a and b have nonempty fixed point sets, which we denote F_a and F_b . If $F_a \cap F_b \neq \emptyset$, then choosing $p \in F_a \cap F_b$ we have that both a and b stabilize the line $\mathbf{R} \times \{p\} \subset \mathbf{R} \times T$.

So suppose that $F_a \cap F_b = \emptyset$. As in [13], we have that ab is hyperbolic in its action on T , stabilizing a line ℓ which intersects both F_a and F_b . We claim that a stabilizes ℓ . For otherwise, consider the line $a\ell$. This is stabilized by the element $u = a(ab)a^{-1}$. If $a\ell \neq \ell$, then we obtain that $(ab)^{\pm 1}$ and $u^{\pm 1}$ freely generate a free semigroup by Proposition 10, contradicting our assumption. Similarly, we see that b stabilizes ℓ as well. Thus $\langle a, b \rangle$ stabilizes the flat $\mathbf{R} \times \ell \subset \mathbf{R} \times T$, as required.

We now consider Case II, so that a and b are subparallel to one another. Note that since an axis for a is parallel to a hyperplane (in $sk(b)$), then all the hyperplanes in $sk(a)$ are disjoint. Similarly all the hyperplanes in $sk(b)$ are disjoint. Note also every hyperplane in $sk(a)$ crosses every hyperplane in $sk(b)$, so that they determine a flat $E = Y_a \cap Y_b$. Moreover since b is parallel to one of the hyperplanes in $sk(a)$, it is parallel or peripheral to all hyperplanes in $sk(a)$. But then by Proposition 14, we have that for all disjoint parallel sets $P \in sk(a)$, we have $bP \subset sk(a)$. Thus $b sk(a) \subset sk(a)$. By the same argument, we obtain $b^{-1} sk(a) \subset sk(a)$, so that $b(sk(a)) = sk(a)$.

Similarly, we have that $a(sk(b)) = sk(b)$. We thus have that $\langle a, b \rangle$ stabilizes the flat E .

Finally, we consider Case III. In this case there exists a disjoint skewer set P for a , so that b is parallel to P . However, since a is not subparallel to b , a also skewers every element in $sk(b)$. Since the hyperplanes in $sk(b)$ all intersect the hyperplanes in P , we have that $sk(a)$ has crossing hyperplanes. It follows that the parallel set Y_a for a is of the form $Y_a = E \times \{\text{point}\}$. It is also easy to see that b stabilizes E , so that $\langle a, b \rangle$ stabilizes E . \square

We are now ready to prove Theorem 1, which we restate here for convenience.

Theorem. *Let F be a finite collection of hyperbolic automorphisms of a CAT(0) square complex. Then either*

- (1) *there exists a pair of elements in $a, b \in F$ and a pair words of length at most 7 in a and b which freely generate a free semigroup, or*

- (2) *there exists a flat (of dimension 1 or 2) in X stabilized by all elements of F .*

Proof. Consider $S = \{s_1, s_2, \dots, s_n\}$. Each of the pairs $\{s_i, s_j\}$ satisfy one of the cases I, II, or III, above.

If there exists a pair of type III, without loss of generality, assume that is the pair $\{s_1, s_2\}$, with s_2 subparallel to s_1 and s_1 not subparallel to s_2 . In this case, the parallel set $Y_{s_1} = E \times \{\text{point}\}$. In this case, for every other s_i , we have that the pair s_1, s_i is either of type I or III. In either case, we obtain that s_i stabilizes E and we are done.

So we suppose that no pair $\{s_i, s_j\}$ is of type III. Suppose, that there exists a pair, say $\{s_1, s_2\}$, which is of type II. Let E be the flat in X on which $\langle s_1, s_2 \rangle$ acts. For any other s_i , we have that the pairs $\{s_1, s_i\}$ and $\{s_2, s_i\}$ is of type I or II. It cannot be that both pairs are of type I since $sk(s_1) \cap sk(s_2) = \emptyset$. Also, it cannot be that s_i is subparallel to both s_1 and s_2 , for otherwise ℓ_{s_i} would be parallel to hyperplanes in $sk(s_1)$ and in $sk(s_2)$, but every hyperplane in $sk(s_1)$ crosses every hyperplane in $sk(s_2)$ in a single point. Thus a line cannot be parallel to a hyperplane in $sk(s_1)$ and a hyperplane in $sk(s_2)$. It follows that, without loss of generality, s_i is subparallel to s_1 and $sk(s_i) = sk(s_2)$. It then follows that s_i stabilizes E .

Finally, suppose that all the pairs s_i, s_j are of type I. Thus $sk(s_i) = sk(s_j)$ for all i, j . Thus G stabilizes $Y = E \times T = E_{s_i} \times T_{s_i}$. If E contains squares, then T is trivial and s_i stabilizes E as required. So suppose that $Y = \mathbf{R} \times T$, and each s_i acts “vertically”. That is, s_i acts by translation along \mathbf{R} and has a fixed point in T .

We now examine the action of G on T . Let F_i denote the fixed set of a_i . If for each pair i, j , $F_i \cap F_j \neq \emptyset$, then by a standard result, $X_n = \bigcap_{i=1}^n F_i \neq \emptyset$. Choose a vertex $p_n \in X_n$. Then $H_n = \langle s_1, \dots, s_n \rangle$ acts on $\ell_n = \mathbf{R} \times p_n$ by translations. Thus H_n stabilizes a flat in X .

So suppose that there exists a pair, say F_1 and F_2 , such that $F_1 \cap F_2 = \emptyset$. In this case, as in the proof of Proposition 16, there exist a line $\ell \subset T$ on which $\langle s_1, s_2 \rangle$ acts as a dihedral group. As in the proof of Proposition 16, we also obtain that for every i , s_i stabilizes ℓ . Thus G stabilizes ℓ , and therefore the flat $\mathbf{R} \times \ell$, as required. \square

Corollary 2 now follows from the Main Theorem since when the action of a group is free, stabilizing a flat implies the group is virtually abelian, by the Bieberbach Theorem.

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